Neutrino Quantum Kinetic Equations: 
the Collision Term

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Overview

➢ **Motivation**: Impact of neutrino interactions in supernovae, BH accretion-discs, neutron-star mergers

➢ **Background**: Review QKEs

➢ **Collision terms**: Results of work in progress
Motivation
Good description of neutrino interactions

- Neutrino interactions impact abundance of heavy elements in neutrino driven winds in supernovae, BH accretion-discs, neutron-star mergers

- Needed for complete description of neutrino transport in early universe, core collapse supernovae and compact mergers

- Quantum Kinetic Equations (QKE): evolution of ensemble of neutrinos in hot dense media

- Closed Time Path formalism for non-equilibrium QFT
Why QKE?

- Account for kinetic, flavor and spin degrees of freedom: study interaction of all flavors of neutrinos with electrons, protons, neutrons

- More detailed than mean field approach

- Anisotropic regions: spin-flip yields
  - Neutrino-antineutrino transformation for Majorana neutrinos
  - Active-sterile transformation for Dirac neutrino
  - QKEs depend on absolute mass scale

Background
Effective interactions

Assume neutrino energy below electroweak scale (<< 100 GeV), effective Lagrangians after integrating out $W,Z$:

$$\mathcal{L}_{\nu\nu} = -\frac{G_F}{\sqrt{2}} \bar{\nu} \gamma_\mu P_L \nu \bar{\nu} \gamma^\mu P_L \nu,$$

$$\mathcal{L}_{\nu e} = -2\sqrt{2}G_F \left( \bar{\nu} \gamma_\mu P_L Y_e L \nu \bar{e} \gamma^\mu P_L e + \bar{\nu} \gamma_\mu P_L Y_e R \nu \bar{e} \gamma^\mu P_R e \right)$$

$$\mathcal{L}_{\nu N} = -\sqrt{2}G_F \sum_{N=p,n} \bar{\nu} \gamma_\mu P_L \nu \bar{N} \gamma^\mu \left( C_V^{(N)} - C_A^{(N)} \gamma_5 \right) N,$$

$$\mathcal{L}_{CC} = -\sqrt{2}G_F \bar{e} \gamma_\mu P_L \nu e \bar{p} \gamma^\mu (1 - g_A \gamma_5) n + \text{h.c.}$$

$$P_{L,R} = (I \mp \gamma_5)/2, \quad \nu = \begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix}.$$
Effective interactions

... with electron couplings

\[ Y_{eL} = \text{diag}\left( \frac{1}{2} + \sin^2 \theta_W, -\frac{1}{2} + \sin^2 \theta_W, -\frac{1}{2} + \sin^2 \theta_W \right) , \]
\[ Y_{eR} = \sin^2 \theta_W \times I . \]

Nucleon couplings:

\[ C_V^{(p)} = \frac{1}{2} - 2 \sin^2 \theta_W , \quad C_V^{(n)} = -\frac{1}{2} , \]
\[ C_A^{(p)} = \frac{g_A}{2} , \quad C_A^{(n)} = -\frac{g_A}{2} , \quad g_A \sim 1.27 \]
Neutrinos in hot/dense medium

ensemble of neutrinos described by incoherent mixture of states

\[
\begin{align*}
\langle a^\dagger_{j,h'}(\vec{k}') a_{i,h}(\vec{k}) \rangle &\propto \delta^3(\vec{k} - \vec{k}') f_{hh'}^{ij}(\vec{k}) \\
\langle b^\dagger_{j,h'}(\vec{k}') b_{i,h}(\vec{k}) \rangle &\propto \delta^3(\vec{k} - \vec{k}') f_{hh'}^{ij}(\vec{k})
\end{align*}
\]

\[2n_f \times 2n_f\] matrix structure: Dirac case, need \( F \) and \( \bar{F} \)

\[
F = \begin{pmatrix}
f_{LL} & f_{LR} \\
\bar{f}_{RL} & f_{RR}
\end{pmatrix} \quad \text{active-sterile coherence}
\]

\[
\bar{F} = \begin{pmatrix}
\bar{f}_{RR} & \bar{f}_{RL} \\
f_{LR} & \bar{f}_{LL}
\end{pmatrix}
\]

Majorana case:

\[
F = \begin{pmatrix}
f \\
\phi^\dagger \\
\phi \\
f^T
\end{pmatrix} \quad \text{neutrino-antineutrino coherence}
\]
Closed Time Path (CTP) formalism

2-point function:

\[ G(x, y) = \langle T_{\text{CTP}} (\Psi(x) \bar{\Psi}(y)) \rangle =: F(x, y) - \frac{i}{2} \rho(x, y) \text{sgn}_{\text{CTP}}(x^0 - y^0) \]

- Time-ordering along closed time path
- Ensemble average

Statistical fct / occupation #:

\[ F(x, y) = \frac{1}{2} \langle [\Psi(x), \bar{\Psi}(y)] \rangle \]

Spectral function:

\[ \rho(x, y) = i \langle \{ \Psi(x), \bar{\Psi}(y) \} \rangle \]

Wigner transform:

\[ F(X, k) = \int d^4r e^{ikr} F(X + \frac{1}{2}r, X - \frac{1}{2}r) \]

Introduction to CTP: see e.g. Calzetta, Hu, PRD 37 (1988) 2878
**Power counting / approximations**

Assume neutrino masses, mass-splitting, matter potentials (induced by forward scattering), and external gradients are much smaller than neutrino energy:

\[ \frac{m_\nu}{E} \sim \frac{\Delta m_\nu}{E} \sim \frac{\Sigma_{\text{forward}}}{E} \sim \frac{\partial X}{E} \sim O(\epsilon) \]

\[ \frac{\Sigma_{\text{inelastic}}}{E} \sim O(\epsilon^2) \]

i.e. assume physical quantities vary slowly on the scale of the neutrino de Broglie wavelength

QKEs include second order effects \( O(\epsilon^2) \)

*details: Vlasenko, Fuller, Cirigliano, *PRD* **89** (2014) 105004*
For ultra-relativistic neutrinos, it is useful to express all Lorentz tensors in terms of a basis formed by two light-like four-vectors and two transverse four-vectors:

\[ \hat{\kappa}^{\mu}(p) = (\text{sgn}(p^0), \hat{p}), \quad \hat{\kappa}'^{\mu}(p) = (\text{sgn}(p^0), -\hat{p}), \quad \hat{x}_{1,2}(p), \]

\[ \hat{x}^{\pm} = \hat{x}_1 \pm i\hat{x}_2, \quad \hat{\kappa} \cdot \hat{\kappa}' = 2 = -\hat{x}^+ \cdot \hat{x}^- \]

The four independent spinor components of the Wigner Transform of the neutrino statistical two-point function are:

\[ F_{L,R} = \frac{1}{4} \text{Tr}\left(\gamma_{\mu} P_{L,R} F(p, x)\right) \hat{\kappa}^{\mu} \]

\[ \Phi^{(\dagger)} = \mp \frac{i}{16} \text{Tr}\left(\sigma_{\mu\nu} P_{L/R} F(p, x)\right)(\hat{\kappa} \land \hat{x}^{\pm})^{\mu\nu} e^{\pm i\varphi} \]

These can be collected in a \(2n_f \times 2n_f\) matrix.
Quantum Kinetic Equations (QKE)

\[ iDF = [H, F] + iC \]

- Generalized "Vlasov" term
- Coherent evolution, generalizes MSW
- Collision term

\[ H = \begin{pmatrix} H_R & H_m \\ H_m^\dagger & H_L \end{pmatrix}, \quad H_m \text{ depends linearly on the neutrino mass} \]

Spin flip sensitive to absolute mass scale!

Details: Vlasenko, Fuller, Cirigliano, *PRD* 89 (2014) 105004;
Vlasenko, Fuller, Cirigliano, arXiv:1406.6724
Collision term

\[ C = \frac{1}{2} \{ \Pi^+, F \} - \frac{1}{2} \{ \Pi^-, I - F \} \]

\[ \Pi^\pm = \begin{pmatrix} \Pi_R^{\kappa\pm} & 2P^\pm \\ 2P^{\pm \dagger} & \Pi_L^{\kappa\pm} \end{pmatrix} \]

\[ F = \begin{pmatrix} f & \phi \\ \phi^\dagger & f^T \end{pmatrix} \]

The collision term has a non-diagonal matrix structure in both flavor and spin space.
Results

DNB, V. Cirigliano, *in preparation*
Contributions to the collision term

\[ \Pi : \]

- Neutrino-nucleon scattering processes
- Neutrino absorption and emission (charged-current processes)
- Neutrino-electron processes
- Neutrino-neutrino processes

only left topology
Example: NN-scattering

\[ \Pi_{ab}^{\pm}(k) = -2G_F^2 \int \frac{d^4q_1}{(2\pi)^8} \frac{d^4q_2}{(2\pi)^8} \frac{d^4q_3}{(2\pi)^8} \delta^{(4)}(k - q_3 - q_1 + q_2) \]

\[ \times \sum_{N=n,p} \left\{ \gamma_\mu(P_L - P_R) G_{ab}^{(\nu)^\pm(q_3)} \gamma_\nu(P_L - P_R) \right\} \]

\[ \times \text{Tr} \left[ \Gamma_N^{\nu} G^{(N)^\mp(q_2)} \Gamma_N^{\mu} G^{(N)^\pm(q_1)} \right] \]

\[ G^{(N)^+(p)} = 2\pi\delta(p^2 - m_N^2)(\gamma + m_N) \left[ \theta(p^0)(1 - f(\bar{p})) - \theta(-p^0)f(-\bar{p}) \right] \]

Neglect neutrino mass in these expressions because the collision term is already second order \( O(\epsilon^2) \)

12 integrals and 7 delta functions
General expressions: amplitudes

Lorentz projections:

$\Pi^\pm(k) = \begin{pmatrix} \Pi^\kappa_R(k) & 2P^\pm(k) \\ 2P^{\pm\dagger}(k) & \Pi^\kappa_L(k) \end{pmatrix}$

$= \frac{1}{2} \int d^4q_3 \begin{pmatrix} |A_-(q_3, k)|^2 (\bar{G}^L_V)^{\pm}(q_3) & A_+^\dagger(q_3, k)A_+(q_3, k)\Phi^{\pm}(q_3) \\ A_+^\dagger(q_3, k)A_-(q_3, k)\Phi^{\pm\dagger}(q_3) & |A_+(q_3, k)|^2 (\bar{G}^R_V)^{\pm}(q_3) \end{pmatrix}$

- Can be written in terms of (square modulus of) amplitudes

- Generalizes earlier studies

Limit of $k\sim q_3$: compares to old results of L. Stodolski *PRD* 36 (1987) 2273
General expressions

\[
\Pi = \frac{1}{2} \begin{pmatrix}
|A_-|^2 \bar{G}^L_V & A_+^\dagger A_+ \Phi \\
A_+^\dagger A_- \Phi^\dagger & |A_+|^2 \bar{G}^R_V
\end{pmatrix}
\]

Majorana neutrinos:

\[
P_{L/R} \psi(x) = \int \frac{d^3p}{2E(2\pi)^3} \left( u(p, \mp) a(p, \mp) e^{-i p x} + v(p, \pm) a^\dagger(p, \pm) e^{i p x} \right)
\]

\[
A = A_+ + A_- , \quad A_{\pm}(q, p) = \pm \bar{u}(q, \pm) \gamma^\mu u(p, \pm) N_\mu
\]

\[
u(p, \pm) \bar{u}(p, \pm) = \gamma^0 P_L R , \quad u(p, \pm) \bar{u}(p, \mp) = \pm \frac{i}{4} E e^{\pm i \varphi} (\hat{k} \wedge \hat{x}^\pm)_{\mu\nu} \sigma^{\mu\nu}
\]
Neutrino-neutrino interactions

“Wedges” appear also in the diagonal because of the neutrino “target”

Always appear together with off-diagonal statistical fcts $\phi$

Up to four (instead of two) wedges can appear in the off-diagonal (drop out upon integrating the azimuthal angles in the special geometric cases we consider later)

Will be interesting to plug collision-terms into QKEs numerically, but will need simplifying assumptions to be feasible
Approximations for supernovae

Assuming spherical symmetry (in position space; “bulb model”), isotropic emittance of neutrinos, and time independence:
all statistical functions (being Wigner transforms) depend only on
\[ |\vec{k}|, \theta_k, |\vec{x}| \]
Therefore can explicitly integrate over all \( \varphi_k \)
Initially, have a total number of 9 integrals and 4 delta functions;
integrating the azimuthal angles (in k-space) leaves us with 6 integrals and 3 delta functions.
Example: NN-scattering

\[
C = \begin{pmatrix} C & C_{\phi} \\ C_{\phi}^\dagger & C_T \end{pmatrix}
\]

neglecting anti-nucleons:

\[
C = -G_F^2 \int \frac{r_1^3 r_2^3 r_3^3 dr_1 r_2 r_3}{4(2\pi)^4 E_1 E_2 E_3} \frac{d(c_{s1} - c_{s3})}{4(2\pi)^4 E_1 E_2 E_3} \delta(E_k - E_3 - E_1 + E_2) \delta(r_k - r_3 - r_1 + r_2)
\]

\[
\times \delta(c_{s_k} - c_{s3} - c_{s1} + c_{s2}) \left[ \left\{ (1 - f_{N,1}) f_{N,2} (1 - f_3), f \right\} - f \leftrightarrow (1 - f) \right]
\]

\[
\times \left( (C_V - C_A)^2 \left( \frac{E_1}{r_1} - c_{s3} c_{s1} \right) \left( \frac{E_2}{r_2} - c_{s_k} c_{s2} \right) - \frac{m_N^2}{r_1 r_2} (C_V^2 - C_A^2) (1 - c_{s_k} c_{s3}) \right)
\]

\[
+ (C_V + C_A)^2 \left( \frac{E_2}{r_2} - c_{s3} c_{s2} \right) \left( \frac{E_1}{r_1} - c_{s_k} c_{s1} \right) \frac{E_3}{r_3}
\]

\[
+ 8(C_V^2 + C_A^2) \cos^2\left( \frac{\theta_k}{2} \right) \cos^2\left( \frac{\theta_3}{2} \right) \sin^2\left( \frac{\theta_1}{2} \right) \sin^2\left( \frac{\theta_2}{2} \right) \left((f_{N,2} - f_{N,1}) \phi_3 \phi_3^\dagger + \text{h.c.} \right)
\]

3-dim. Integrals left
Approximations for the early universe

- Assume all statistical functions depend only on the absolute values of the momenta (not their angles), spin coherence disappears.

- Therefore can explicitly integrate over all angles (e.g. following techniques of Dolgov, Hansen & Semikoz 1997).

- Initially, have a total number of 9 integrals and 4 delta functions; integrating all angles (in k-space) leaves us with 3 integrals and 1 delta function.

- Represents multi-flavor generalization of previous work.

**Future: solve these 2-dim. Integrals numerically?**
Example: neutrino-neutrino processes

spin coherence disappears in early universe, therefore:

\[
C = -\frac{G_F^2}{E_k^2} \int \frac{dE_1 dE_2 dE_3}{2\pi^3} \left( (E_1 E_3 D_2(E_1, E_3; E_2, E_k) + D_3(E_1, E_2, E_3, E_k) \right) + E_2 E_k D_2(E_2, E_k; E_1, E_3) + E_1 E_2 E_3 E_k D_1(E_1, E_2, E_3, E_k) \right) \times
\]

\[
\times \left( \delta(E_k - E_3 - E_1 + E_2) \left\{ \left( \text{tr}((1 - f_1)f_2) + (1 - f_1)f_2 \right)(1 - f_3), f \right\} + \delta(E_k - E_3 + E_1 - E_2) \left\{ \left( \text{tr}(\bar{f}_1(1 - \bar{f}_2)) + \bar{f}_1(1 - \bar{f}_2) \right)(1 - f_3), f \right\} + \delta(E_k + E_3 - E_1 - E_2) \left\{ \left( \text{tr}((1 - f_1)(1 - \bar{f}_2)) + (1 - f_1)(1 - \bar{f}_2) \right)\bar{f}_3, f \right\} \right) \right)
\]

\[
- f \leftrightarrow (1 - f)
\]

where \( D_i \) are polynomials in \( E_i \) (Dolgov, Hansen, Semikoz 1997)

multi-flavor generalization
Conclusion and Outlook

✔ Introduced and motivated the concept of QKEs

✔ Presented results for collision terms in the Majorana case

✔ Remaining integrals should be solvable numerically

✔ To do: generalize to Dirac neutrinos
References

1. D. N. Blaschke, V. Cirigliano, et al., *in preparation*

